

A note on isospin violation in $P_{\ell 2(\gamma)}$ decays

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Abstract

We discuss the size of isospin violating effects of both electromagnetic ($e \neq 0$) and strong ($m_u \neq m_d$) origin in $P_{\ell 2(\gamma)}$ decays. Our results are relevant for the extraction of $|V_{us}/V_{ud}|$ from the measured ratio of the $K_{\mu 2(\gamma)}$ and $\pi_{\mu 2(\gamma)}$ decay widths combined with F_K/F_π obtained in lattice calculations in the isospin limit. We point out that strong isospin-breaking corrections, neglected in previous studies, enter at the same level as the electromagnetic effects. The updated value for the ratio of CKM elements reads $|V_{us}/V_{ud}| = 0.2316(12)$.

1 Introduction

The purely leptonic decays of light pseudoscalar mesons $P \rightarrow \ell \nu_\ell$ ($P^\pm = \pi^\pm, K^\pm$) are theoretically extremely clean. Neglecting electroweak corrections, the hadronic physics is encoded in one single parameter, the meson decay constant F_P . Using chiral perturbation theory and lattice QCD, $P \rightarrow \ell \nu_\ell$ decay rates can be predicted with high accuracy and provide non-trivial tests of the standard model imposing stringent constraints on new physics.

With the advent of high precision calculations of the ratio of decay constants F_K/F_π in lattice QCD, it was realized [1] that the combination of F_K/F_π and the ratio of measured decay widths $\Gamma_{K_{\mu 2}(\gamma)}/\Gamma_{\pi_{\mu 2}(\gamma)}$ leads to a clean and competitive determination of the ratio $|V_{us}/V_{ud}|$ of Cabibbo-Kobayashi-Maskawa (CKM) [2, 3] matrix elements. Implementing the programme proposed in Ref. [1] requires theoretical understanding of: (i) the long-distance electromagnetic (EM) corrections to $P \rightarrow \ell \nu_\ell$; (ii) the strong isospin breaking correction that connects F_K/F_π (calculated within lattice QCD in the isospin limit $m_u = m_d$) to F_{K^\pm}/F_{π^\pm} that enters the ratio of $K_{\mu 2}(\gamma)$ to $\pi_{\mu 2}(\gamma)$ decay widths.

The EM corrections have been known for some time [4, 5, 6, 7]. The analysis of Ref. [1] relied on the model-dependent treatment of Ref. [7] for the long-distance contribution to these corrections. The long-distance EM corrections to $P \rightarrow \ell \nu_\ell(\gamma)$ have also been analyzed in the model-independent framework of Chiral Perturbation Theory (ChPT) [8, 9, 10]. It was noted in Ref. [8] that the radiative corrections to the ratio $\Gamma_{K_{\mu 2}(\gamma)}/\Gamma_{\pi_{\mu 2}(\gamma)}$ are uniquely determined to lowest non-trivial order in the chiral expansion and the final numerical results have been reported in Refs. [11, 12]. So far, strong isospin breaking corrections $\propto (m_u - m_d)$ have been ignored in the extraction of $|V_{us}/V_{ud}|$. In view of the increasing precision of lattice results on F_K/F_π , also these effects should be taken into account and it is the purpose of this letter to fill this gap.

In Section 2 we perform our analysis of both electromagnetic and strong isospin violation to $\Gamma_{P_{\ell 2}(\gamma)}$ in the unified framework of ChPT. In Section 3 we update the phenomenological determination of $|V_{us}/V_{ud}|$ and the global fit to $|V_{ud}|$ and $|V_{us}|$ [12] to reflect the new strong isospin breaking correction. We present our conclusions in Section 4.

2 Isospin violation in $\pi \rightarrow \ell \nu_\ell(\gamma)$ and $K \rightarrow \ell \nu_\ell(\gamma)$

The fully photon-inclusive decay width

$$\Gamma_{P_{\ell 2}(\gamma)} \equiv \Gamma(P \rightarrow \ell \nu_\ell) + \Gamma(P \rightarrow \ell \nu_\ell \gamma) , \quad P = \pi^\pm, K^\pm , \quad (2.1)$$

can be written in the form [4, 9]

$$\Gamma_{P_{\ell 2}(\gamma)} = \frac{G_F^2 |V_P|^2 F_P^2}{4\pi} M_P m_\ell^2 \left(1 - \frac{m_\ell^2}{M_P^2} \right)^2 S_{\text{EW}} \left(1 + \delta_{\text{EM}}^P \right) , \quad V_{\pi^\pm} = V_{ud} , V_{K^\pm} = V_{us} . \quad (2.2)$$

G_F is the Fermi constant extracted from muon decay and F_{π^\pm} (F_{K^\pm}) is the decay constant of the charged pion (kaon) in pure QCD¹. $S_{\text{EW}} = 1.0232$ is the universal electroweak short-distance enhancement factor [4, 5, 6] appearing in semileptonic decays. The long-distance electromagnetic corrections are described by [4]

$$\delta_{\text{EM}}^P = \frac{\alpha}{\pi} \left(F(m_\ell/M_P) + \frac{3}{4} \ln \frac{M_P^2}{M_\rho^2} - C_1^P + \dots \right). \quad (2.3)$$

The one-loop function $F(x)$ is related to the universal long-distance electromagnetic corrections (limit of point-like meson), its explicit form can be found in Ref. [4]. The dimensionless constant C_1^P parametrizes electromagnetic effects due to hadronic structure. The dots refer to further structure dependent terms [9] arising at higher orders in the chiral expansion.

The constants C_1^π and C_1^K were calculated in chiral perturbation theory with virtual photons and leptons to order $e^2 p^2$ [8]:

$$C_1^\pi = -\tilde{E}^r(M_\rho) + \frac{Z}{4} \left(3 + 2 \ln \frac{M_\pi^2}{M_\rho^2} + \ln \frac{M_K^2}{M_\rho^2} \right), \quad (2.4)$$

$$C_1^K = -\tilde{E}^r(M_\rho) + \frac{Z}{4} \left(3 + 2 \ln \frac{M_K^2}{M_\rho^2} + \ln \frac{M_\pi^2}{M_\rho^2} \right). \quad (2.5)$$

The parameter Z , appearing in the chiral Lagrangian of order $e^2 p^0$, can be expressed in terms of the measured pion masses, the fine-structure constant and the pion decay constant:

$$Z = \frac{M_{\pi^\pm}^2 - M_{\pi^0}^2}{8\pi\alpha F_\pi^2} \simeq 0.8. \quad (2.6)$$

Both C_1^π and C_1^K depend on the same combination of electromagnetic low-energy couplings,

$$\tilde{E}^r = \frac{1}{2} + 4\pi^2 \left(\frac{8}{3} K_1^r + \frac{8}{3} K_2^r + \frac{20}{9} K_5^r + \frac{20}{9} K_6^r - \frac{4}{3} X_1^r - 4X_2^r + 4X_3^r - \tilde{X}_6^{\text{phys}} \right). \quad (2.7)$$

The K_i were defined in Ref. [15] and the X_i in Ref. [8]. Note that we have pulled out the short-distance enhancement factor in Eq. (2.2) keeping only the residual long-distance part $\tilde{X}_6^{\text{phys}}$ [16] in Eq. (2.7). The coupling constants entering in Eq. (2.7) have been estimated in Refs. [17] and [16] using large- N_c methods, giving

$$C_1^\pi = -2.4(5), \quad C_1^K = -1.9(5). \quad (2.8)$$

¹In the case of the pion, the distinction between the charged and the neutral decay constant is a tiny effect of order $(m_d - m_u)^2$, whereas F_{K^\pm}/F_{K^0} is of the order $m_d - m_u$ [13]. For a discussion of some subtle points in the separation of QCD and electromagnetism, the reader is referred to Refs. [8], [14], and [10]. Here we simply recall that the scale ambiguity inherent to the separation of QCD and QED effects in the meson decay constants in the chiral limit has been shown to be about 8 keV [10], and thus negligible at the current level of precision.

The errors given here are based on naive power counting of unknown contributions arising at order $e^2 p^4$.

For the ratio of decay widths we find

$$\frac{\Gamma_{K\ell 2(\gamma)}}{\Gamma_{\pi\ell 2(\gamma)}} = \frac{|V_{us}|^2}{|V_{ud}|^2} \frac{F_K^2}{F_\pi^2} \frac{M_{K^\pm} (1 - m_\ell^2/M_{K^\pm}^2)^2}{M_{\pi^\pm} (1 - m_\ell^2/M_{\pi^\pm}^2)^2} (1 + \delta_{\text{EM}}) , \quad (2.9)$$

where

$$\begin{aligned} \delta_{\text{EM}} &= \delta_{\text{EM}}^K - \delta_{\text{EM}}^\pi \\ &= \frac{\alpha}{\pi} \left(F(m_\ell/M_K) - F(m_\ell/M_\pi) + \frac{3}{4} \ln \frac{M_K^2}{M_\pi^2} - C_1^K + C_1^\pi \right) \\ &= \frac{\alpha}{\pi} \left(F(m_\ell/M_K) - F(m_\ell/M_\pi) + \frac{3-Z}{4} \ln \frac{M_K^2}{M_\pi^2} \right) \\ &= -0.0069(17) . \end{aligned} \quad (2.10)$$

The important feature of the above result is that the electromagnetic low-energy coupling \tilde{E}^r cancels out in the ratio [8]. To lowest non-trivial order in the chiral expansion, the electromagnetic correction δ_{EM} is uniquely determined in terms of the fine-structure constant and masses, demonstrating the power of effective field theory methods. The quoted 25% uncertainty is an estimate of corrections arising to higher order in the chiral expansion. Note also that in Eq. (2.9) the ratio F_{K^\pm}/F_{π^\pm} is the full QCD quantity including strong isospin breaking due to $m_u \neq m_d$.

With experimental measurements of the $\pi_{\ell 2(\gamma)}$ and $K_{\ell 2(\gamma)}$ decay rates and the precise knowledge of the radiative corrections given in Eq. (2.10), Eq. (2.9) can be used to obtain the value of the ratio

$$\frac{|V_{us}|}{|V_{ud}|} \frac{F_{K^\pm}}{F_{\pi^\pm}} = \left(\frac{\Gamma_{K\ell 2(\gamma)}}{\Gamma_{\pi\ell 2(\gamma)}} \frac{M_{\pi^\pm}}{M_{K^\pm}} \right)^{1/2} \frac{1 - m_\ell^2/M_{\pi^\pm}^2}{1 - m_\ell^2/M_{K^\pm}^2} (1 - \delta_{\text{EM}}/2) . \quad (2.11)$$

It was suggested in Ref. [1] to combine this result with lattice data on the ratio of the kaon and pion decay constant to determine $|V_{us}/V_{ud}|$. Finally, using also experimental input for $|V_{ud}|$, a value for $|V_{us}|$ can be obtained in this way.

As lattice calculations [18, 19, 20, 21, 22, 23, 24] are usually still performed in the limit of equal light quark masses ($m_u = m_d$), it is convenient to rewrite Eq. (2.9) in the form

$$\frac{\Gamma_{K\ell 2(\gamma)}}{\Gamma_{\pi\ell 2(\gamma)}} = \frac{|V_{us}|^2}{|V_{ud}|^2} \frac{F_K^2}{F_\pi^2} \frac{M_{K^\pm} (1 - m_\ell^2/M_{K^\pm}^2)^2}{M_{\pi^\pm} (1 - m_\ell^2/M_{\pi^\pm}^2)^2} (1 + \delta_{\text{EM}} + \delta_{\text{SU}(2)}) , \quad (2.12)$$

where F_π and F_K denote the decay constants in the isospin limit and

$$\delta_{\text{SU}(2)} = \left(\frac{F_{K^\pm} F_\pi}{F_{\pi^\pm} F_K} \right)^2 - 1 . \quad (2.13)$$

The isospin symmetry limit is defined by $m_u = m_d$, $e = 0$ and an appropriate choice of the isospin symmetric meson masses M_π , M_K . To the order we are working here, it

is sufficient to adopt the following convention: M_π and M_K are related to the measured values of the masses by

$$M_\pi^2 = M_{\pi^0}^2, \quad M_K^2 = \frac{1}{2} (M_{K^\pm}^2 + M_{K^0}^2 - M_{\pi^\pm}^2 + M_{\pi^0}^2), \quad (2.14)$$

as the leading order isospin violating contributions of strong and electromagnetic origin cancel in these expressions (see e.g. Ref [8]). For a fully consistent application of the isospin violating corrections calculated in this work, the lattice QCD results should be extrapolated to the isospin-limit “physical” meson masses M_K and M_π .

The chiral expansion of F_{π^\pm} and F_{K^\pm} up to the order p^4 , $(m_d - m_u)p^2$ is given by [25]

$$F_{\pi^\pm} = F_0 \left\{ 1 + \frac{4}{F_0^2} [L_4^r(\mu)(M_\pi^2 + 2M_K^2) + L_5^r(\mu)M_\pi^2] - \frac{1}{2(4\pi)^2 F_0^2} \left[2M_\pi^2 \ln \frac{M_\pi^2}{\mu^2} + M_K^2 \ln \frac{M_K^2}{\mu^2} \right] \right\}, \quad (2.15)$$

$$F_{K^\pm} = F_0 \left\{ 1 + \frac{4}{F_0^2} [L_4^r(\mu)(M_\pi^2 + 2M_K^2) + L_5^r(\mu)M_K^2] - \frac{1}{8(4\pi)^2 F_0^2} \left[3M_\pi^2 \ln \frac{M_\pi^2}{\mu^2} + 6M_K^2 \ln \frac{M_K^2}{\mu^2} + 3M_\eta^2 \ln \frac{M_\eta^2}{\mu^2} \right] - \frac{8\sqrt{3}\varepsilon}{3F_0^2} L_5^r(\mu)(M_K^2 - M_\pi^2) - \frac{\sqrt{3}\varepsilon}{4(4\pi)^2 F_0^2} \left[M_\pi^2 \ln \frac{M_\pi^2}{\mu^2} - M_\eta^2 \ln \frac{M_\eta^2}{\mu^2} - \frac{2}{3}(M_K^2 - M_\pi^2) \left(\ln \frac{M_K^2}{\mu^2} + 1 \right) \right] \right\}. \quad (2.16)$$

F_0 is the pion decay constant in the limit of chiral SU(3) and M_π and M_K denote the isospin limits of the pion mass and the kaon mass, respectively, as discussed above. To lowest order, they can be expressed in terms of the quark masses by

$$M_\pi^2 = 2B_0\hat{m}, \quad M_K^2 = B_0(m_s + \hat{m}), \quad \hat{m} = \frac{1}{2}(m_u + m_d), \quad (2.17)$$

where B_0 is related to the quark condensate. To the same order, M_η is related to M_K and M_π by the GMO relation

$$M_\eta^2 = \frac{4}{3}B_0 \left(m_s + \frac{\hat{m}}{2} \right) = \frac{4}{3}M_K^2 - \frac{1}{3}M_\pi^2. \quad (2.18)$$

The parameter ε of strong isospin breaking (which coincides with the lowest order π^0 - η mixing angle) is given by

$$\varepsilon = \frac{\sqrt{3}}{4} \frac{m_d - m_u}{m_s - \hat{m}}. \quad (2.19)$$

Taking $\varepsilon = 0$ in Eqs. (2.15) and (2.16) reproduces the results given in Ref. [13] in the isospin limit. The correction parameter $\delta_{\text{SU}(2)}$ can be read off as

$$\begin{aligned} \delta_{\text{SU}(2)} = & -\frac{16\sqrt{3}}{3F_0^2}\varepsilon L_5^r(\mu)(M_K^2 - M_\pi^2) \\ & -\frac{\sqrt{3}}{2(4\pi)^2F_0^2}\varepsilon \left[M_\pi^2 \ln \frac{M_\pi^2}{\mu^2} - M_\eta^2 \ln \frac{M_\eta^2}{\mu^2} - \frac{2}{3}(M_K^2 - M_\pi^2) \left(\ln \frac{M_K^2}{\mu^2} + 1 \right) \right]. \end{aligned} \quad (2.20)$$

Expressing L_5^r in terms of the *isospin limit* ratio F_K/F_π , $\delta_{\text{SU}(2)}$ can be recast into the following compact form:

$$\delta_{\text{SU}(2)} = \sqrt{3}\varepsilon \left[-\frac{4}{3}(F_K/F_\pi - 1) + \frac{1}{3(4\pi)^2F_0^2} \left(M_K^2 - M_\pi^2 - M_\pi^2 \ln \frac{M_K^2}{M_\pi^2} \right) \right]. \quad (2.21)$$

With the FLAG [26] averages of $N_f = 2 + 1$ lattice calculations,

$$\varepsilon = \frac{\sqrt{3}}{4R} = 0.0116(13), \quad F_K/F_\pi = 1.193(6), \quad (2.22)$$

we obtain

$$\delta_{\text{SU}(2)} = -0.0043(5)(11)_{\text{higher orders}}, \quad (2.23)$$

where we have estimated the uncertainty due to higher order² corrections in the chiral expansion to be at a level of 25%. We note that the strong isospin-breaking correction $\delta_{\text{SU}(2)}$ is of the same order of magnitude as the electromagnetic correction δ_{EM} in Eq. (2.10) and should not be neglected in the extraction of the ratio $|V_{us}/V_{ud}|$.

3 Update of $|V_{us}/V_{ud}|$ and fit to $|V_{ud}|, |V_{us}|$

We start from the basic formula

$$\frac{|V_{us}|}{|V_{ud}|} \frac{F_K}{F_\pi} = \left(\frac{\Gamma_{K\ell 2(\gamma)} M_{\pi^\pm}}{\Gamma_{\pi\ell 2(\gamma)} M_{K^\pm}} \right)^{1/2} \frac{1 - m_\ell^2/M_{\pi^\pm}^2}{1 - m_\ell^2/M_{K^\pm}^2} (1 - \delta_{\text{EM}}/2 - \delta_{\text{SU}(2)}/2). \quad (3.1)$$

Putting together the results on isospin breaking obtained in the previous section and the experimental values for the leptonic widths of the pion $\Gamma_{\pi\mu 2(\gamma)} = 38.408(7) (\mu s)^{-1}$ [28] and the kaon $\Gamma_{K\mu 2(\gamma)} = 51.25(16) (\mu s)^{-1}$ [11] we obtain:

$$\begin{aligned} \frac{|V_{us}|F_K}{|V_{ud}|F_\pi} &= 0.23922(25) \times \left(\frac{\Gamma_{K\ell 2(\gamma)}}{\Gamma_{\pi\ell 2(\gamma)}} \right)^{1/2} \\ &= 0.2763(5). \end{aligned} \quad (3.2)$$

² An analysis of isospin violation beyond $O(p^4)$ in ChPT [27] is beyond the scope of this work.

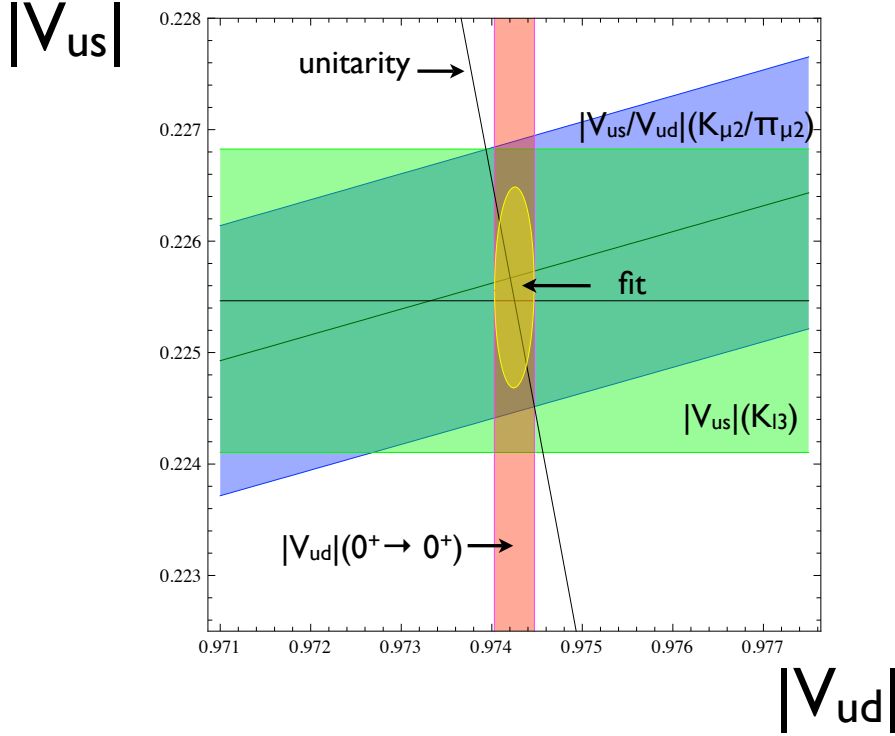


Figure 1: Graphical representation of current status of $|V_{ud}|$, $|V_{us}|$ and corresponding CKM unitarity test. The horizontal band represents the constraint from $K_{\ell 3}$ decays, the thin vertical band the constraint from $0^+ \rightarrow 0^+$ nuclear decays, the oblique band the constraint from $K_{\mu 2}/\pi_{\mu 2}$, and the ellipse is the 1σ fit region.

Finally, taking as reference value for F_K/F_π the FLAG [26] average of $N_f = 2 + 1$ calculations (see Eq. (2.22)) we find for the ratio of the CKM elements:

$$\frac{|V_{us}|}{|V_{ud}|} = 0.2316(12) . \quad (3.3)$$

As discussed in Ref. [12], one can perform a fit to $|V_{ud}|$ and $|V_{us}|$ using as input the values of $|V_{ud}| = 0.97425(22)$ from superallowed nuclear β transitions [29], $|V_{us}/V_{ud}|$ from $K_{\ell 2}/\pi_{\ell 2}$ (Eq. (3.3)), and $|V_{us}|$ from $K_{\ell 3}$ decays. Measurements of $K_{\ell 3}$ decays (BRs, form factors) [12], supplemented with theoretical calculations of electromagnetic and isospin corrections [30, 31] allow one to extract the product $|V_{us}|f_+(0) = 0.2163(5)$ [12]. Combining this with the $N_f = 2 + 1$ lattice result for the $K \rightarrow \pi$ vector form factor $f_+(0) = 0.9599(34)^{+31}_{-47}(14)$ [32] one obtains [26]:

$$|V_{us}|(K_{\ell 3}) = 0.2255(5)_{\text{exp}}(12)_{\text{th}} . \quad (3.4)$$

A fit to the above described input leads to

$$|V_{ud}| = 0.97425(22) , \quad |V_{us}| = 0.2256(9) , \quad (3.5)$$

with $\chi^2/\text{ndf} = 0.012$ and negligible correlations between $|V_{ud}|$ and $|V_{us}|$. Fig. 1 provides a graphical representation of the various constraints in the $|V_{us}| - |V_{ud}|$ plane and the 1σ fit region (in yellow).

The fit values of $|V_{us}|$ and $|V_{ud}|$ (together with the negligible contribution from $|V_{ub}| = 0.00393(36)$ [11]) imply a stringent test of CKM unitarity:

$$\Delta_{\text{CKM}} = |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 - 1 = 0.0001(6) . \quad (3.6)$$

The remarkable agreement with the standard model prediction $\Delta_{\text{CKM}} = 0$ allows one to set strong bounds on the effective scale of dimension-six operators that violate quark-lepton universality of charged current weak interactions [33] and thus parameterize new physics contributions to Δ_{CKM} . The new physics effective scale is constrained to be $\Lambda > 11$ TeV (90% C.L.), a lower bound to be compared to the effective scale of the standard model set by the Fermi constant, $\Lambda_{\text{SM}} = (2\sqrt{2}G_F)^{-1/2} \simeq 174$ GeV. This puts this charged-current quark lepton universality test at the same level as the precision electroweak tests from Z -pole measurements.

4 Conclusions

In this letter we have discussed in detail the effect of electromagnetic and strong isospin breaking relevant for the extraction of $|V_{us}/V_{ud}|$ from the combination of the measured $\Gamma_{K\mu 2(\gamma)}/\Gamma_{\pi\mu 2(\gamma)}$ and the ratio of decay constants F_K/F_π calculated in lattice QCD *in the isospin limit*. We have performed our analysis in the unified framework of ChPT, providing analytic expressions and numerical estimates for the EM and strong isospin breaking corrections δ_{EM} and $\delta_{\text{SU}(2)}$ that appear in the ratio $\Gamma_{K\mu 2(\gamma)}/\Gamma_{\pi\mu 2(\gamma)}$ (see Eq. (2.12)).

The main new result of our work is a compact analytic expression for the correction due to strong isospin breaking (see Eq. (2.21)). This effect, neglected in all previous extractions of $|V_{us}/V_{ud}|$, is comparable in size to the electromagnetic correction:

$$\delta_{\text{EM}} = -0.0069(17) , \quad \delta_{\text{SU}(2)} = -0.0043(12) . \quad (4.1)$$

Using the current experimental results for the decay widths and $F_K/F_\pi = 1.193(6)$ [26], we obtain $|V_{us}/V_{ud}| = 0.2316(12)$, which is about 0.2% above previous determinations that ignored $\delta_{\text{SU}(2)}$.

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